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On N-Widths of Holomorphic Functions of Several Variables

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We consider the classes of holomorphic functions whose radial derivative of order r lies in the unit ball of the Hardy space $H_2(B_n)$ or the Bergman space $A_2(B_n)$. For these classes we calculate the linear and Gel'fand N-widths in $C(S_\rho)$, where S_ρ is the sphere in $\mathbb{C}^n$ of radius $0 < \rho < 1$. Some results are obtained for analogous problems in polydiscs and for 2π-periodic functions of one variable holomorphic in a strip.

INTRODUCTION

Let A be a subset of a normed linear space X. The Kolmogorov N-width is defined by

$$d_N(A, X) := \inf_{X_N} \sup_{x \in A} \inf_{y \in X_N} \|x - y\|,$$

where X_N runs over all N-dimensional subspaces of X. Denote by $\mathscr{L}(H, X)$ the class of all continuous linear operators from H to X, where H and X are normed linear spaces. Let BH be the closed unit ball of H. For $T \in \mathscr{L}(H, X)$ set

$$d_N(T) := d_N(T(BH), X).$$

The linear N-width is given by

$$\lambda_N(A, X) := \inf_{P_N} \sup_{x \in A} \|x - P_N x\|,$$

where P_N runs over all bounded linear operators mapping X into X, whose range has dimension N or less. Assume that $0 \in A$. The Gel'fand N-width is defined by

$$d^N(A, X) := \inf_{X^N} \sup_{x \in A \cap X^N} \|x\|,$$

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where the infimum is taken over all subspaces X^N of X of codimension N. Various properties of these N-widths (and others) may be found in [1].

Let B_n be the unit ball of $\mathbb{C}^n$

$$B_n := \left\{ z := (z_1, \dots, z_n) \in \mathbb{C}^n : |z|^2 := \sum_{k=1}^{n} |z_k|^2 < 1 \right\},$$

and S_ρ the sphere of radius ρ

$$S_\rho := \{ z \in \mathbb{C}^n : |z| = \rho \}$$

(if $\rho = 1$ we write S). The Hardy space $H_p(B_n)$ is the set of holomorphic functions in B_n which satisfy

$$\|f\|_{H_p(B_n)} := \sup_{0<r<1} \left(\int_S |f(rz)|^p \, d\sigma(z) \right)^{1/p} < \infty, \qquad 1 \le p < \infty,$$

$$\|f\|_{H_\infty(B_n)} := \sup_{z \in B_n} |f(z)|,$$

where σ is the probability measure on the sphere S which is invariant with respect to the orthogonal group $O(2n)$. The Bergman space $A_p(B_n)$ is the set of holomorphic functions in B_n which satisfy the condition

$$\|f\|_{A_p(B_n)} := \left(\int_{B_n} |f(z)|^p \, d\nu(z) \right)^{1/p} < \infty,$$

where ν is the normalized Lebesgue measure in B_n ($A_\infty(B_n) = H_\infty(B_n)$).

Let $f(z)$ be a holomorphic function in B_n and

$$f(z) = \sum_{s=0}^{\infty} F_s(z)$$

be a homogeneous decomposition of f. The radial derivative of order r is defined by

$$\mathscr{R}^r f(z) := \sum_{s=r}^{\infty} \frac{s!}{(s-r)!} F_s(z)$$

(for $r = 1$ see [2, Chap. 6]). Let BX be the closed unit ball of a normed linear space X. We denote by $H\mathscr{R}_p^r(B_n)$ and $A\mathscr{R}_p^r(B_n)$ the classes of holomorphic functions in B_n for which $\mathscr{R}^r f$ lie in $BH_p(B_n)$ and $BA_p(B_n)$, respectively.

The exact values of $d_N(H\mathscr{B}_p^r(B_n), L_p(S_p))$ were obtained in [3]. When $n = 1$, $1 \leq q \leq p \leq \infty$ and E is a compact subset of B_1, the values of $d_N(BH_p(B_1), L_q(E))$ were determined in [4] (for $E = S_p$ see also [5]).

The first result for the classes of holomorphic functions concerning the case when $p < q$ appeared in [6] where the values of $d^N(BH_2(B_n), C(S_p))$ and $\lambda_N(BH_2(B_n), C(S_p))$ were obtained (more precisely, for some subsequence of $\mathbb{N}$). The method of proof, as noted by V. M. Tikhomirov, was very similar to the one used in Ismagilov's Theorem [7] (see also [1]). In Section 1 we prove a theorem dual to the Ismagilov Theorem. Using this result, in Section 2 we obtain the values of the linear and Gel'fand N-widths of the classes $H\mathscr{B}_2^r(B_n)$ and $A\mathscr{B}_2^r(B_n)$ in $C(S_p)$.

Section 3 is devoted to analogous problems in polydiscs. Finally in Section 4 we calculate the N-widths of holomorphic functions in the annulus

$$\Delta_R := \{z \in \mathbb{C} : R^{-1} < |z| < R\}, \qquad R > 1,$$

and 2π-periodic functions holomorphic in the strip

$$D_H := \{z \in \mathbb{C} : |\mathrm{Im}\, z| < H\}.$$

1. A Theorem Dual to Ismagilov's Theorem

Let E be a compact set, μ a positive probability measure defined on E and $T \in \mathscr{L}(H, C(E))$. Denote by T_0 the operator T regarded as an operator from H into $L_2(E, \mu)$. Assume that

$$T_0' T_0 \phi_j = \lambda_j \phi_j, \qquad j = 1, 2, \dots,$$

where $\lambda_1 \geq \lambda_2 \geq \cdots > 0$, and that $\phi_1, \phi_2, \dots$ is a complete orthonormal basis for the range of $T_0' T_0$ (a sufficient condition is that T_0 be a compact operator).

Theorem 1. *For T as above*

$$\begin{aligned}\sqrt{\textstyle\sum_{j=N+1}^{\infty} \lambda_j} &\leq d^N(T(BH), C(E)) \\ &= \lambda_N(T(BH), C(E)) \leq \sup_{z \in E} \sqrt{\sum_{j=N+1}^{\infty} |(T\phi_j)(z)|^2}.\end{aligned}$$

Proof. Since $\mathrm{Ker}\, T_0' T_0 = \mathrm{Ker}\, T_0 = \mathrm{Ker}\, T$ we shall assume, without loss of generality, that $\phi_1, \phi_2, \dots$ is a complete orthonormal basis for H. Set

$\psi_j := T\phi_j$. Let us show that for all $z \in E$

$$\sum_{j=1}^{\infty} |\psi_j(z)|^2 \le \|T\|^2 := \left(\sup_{\|h\|_H \le 1} \|Th\|_\infty \right)^2 \tag{1}$$

(we denote by $\|\cdot\|_\infty$ the norm in $C(E)$ and by $\|\cdot\|_H$ the norm in H). Let $z \in E$ and $m \in \mathbb{N}$. Then for $h := \sum_{j=1}^{m} \overline{\psi_j(z)} \phi_j \in H$ we have

$$\|Th\|_\infty = \sup_{s \in E} \left| \sum_{j=1}^{m} \overline{\psi_j(z)} \psi_j(s) \right| \ge \sum_{j=1}^{m} |\psi_j(z)|^2 = \|h\|_H^2.$$

Thus for $h \ne 0$

$$\|h\|_H \le \frac{\|Th\|_\infty}{\|h\|_H} \le \|T\|.$$

Consequently for all $z \in E$ and all $m \in \mathbb{N}$ the inequality

$$\sum_{j=1}^{m} |\psi_j(z)|^2 \le \|T\|^2$$

holds. So (1) is proved.

Set

$$h_z := \sum_{j=1}^{\infty} \overline{\psi_j(z)} \phi_j.$$

It is easy to check that for all $x \in H$ and all $z \in E$

$$(Tx)(z) = (x, h_z)_H.$$

Denote by φ: $E \to H$ the mapping

$$\varphi(z) := h_z.$$

Then

$$\begin{aligned}\int_E (\varphi(z), \varphi(y))_H \overline{\psi_j(y)}\, d\mu(y) &= \int_E (Th_z)(y) \overline{\psi_j(y)}\, d\mu(y) \\ &= (T_0 h_z, T_0 \phi_j)_{L_2(E,\mu)} \\ &= (h_z, T_0^* T_0 \phi_j)_H = \lambda_j \overline{\psi_j(z)}.\end{aligned}$$

Furthermore

$$(\psi_j, \psi_k)_{L_2(E,\mu)} = \lambda_j \delta_{jk}.$$

By the Ismagilov Theorem we obtain

$$\sqrt{\sum_{j=N+1}^{\infty} \lambda_j} \le d_N(T') \le \sup_{z \in E} \sqrt{\sum_{j=N+1}^{\infty} |(T\phi_j)(z)|^2}.$$

From duality

$$d_N(T') = d^N(T) := \inf_{X^N} \sup_{h \in BH \cap X^N} \|Th\|_\infty,$$

where the infimum is taken over all subspaces X^N of H of codimension N. Since H is a Hilbert space

$$d_N(T') = d^N(T(BH), C(E)) = \lambda_N(T(BH), C(E)).$$

The theorem is proved. ∎

COROLLARY 1. *Assume that the conditions of Theorem 1 hold and X_r is any r-dimensional subspace of $C(E)$ such that $X_r \perp T_0(H)$ in $L_2(E, \mu)$. Then*

$$\sqrt{\sum_{j=N+1}^{\infty} \lambda_j} \le d^{N+r}(T(BH) + X_r, C(E)) = \lambda_{N+r}(T(BH) + X_r, C(E))$$

$$\le \sup_{z \in E} \sqrt{\sum_{j=N+1}^{\infty} |(T\phi_j)(z)|^2}.$$

Proof. Let $e_1, \ldots, e_r$ be an orthonormal basis for X_r in $L_2(E, \mu)$. Denote by $H_{r,\varepsilon}$ the Hilbert space of elements $\{f, g\}$, $f \in H$, $g \in X_r$ with inner product

$$(\{f_1, g_1\}, \{f_2, g_2\})_{H_{r,\varepsilon}} := (f_1, f_2)_H + \varepsilon \sum_{j=1}^{r} c_j \bar{d}_j, \qquad \varepsilon > 0,$$

where

$$g_1 = \sum_{j=1}^{r} c_j e_j, \qquad g_2 = \sum_{j=1}^{r} d_j e_j.$$

Put $L\{f, g\} := Tf + g$. Denote by L_0 the operator L as an operator from

$H_{r,\varepsilon}$ into $L_2(E, \mu)$. Then

$$L_0' L_0\{f, g\} = \{T_0' T_0 f, \varepsilon^{-1} g\}.$$

Set

$$\varphi_j := \{0, \varepsilon^{-1/2} e_j\}, \quad j = 1, \dots, r, \quad \varphi_j := \{\phi_{j-r}, 0\}, \quad j = r+1, \dots .$$

The elements $\varphi_1, \varphi_2, \dots$ form a complete orthonormal basis for the range of $L_0' L_0$ and

$$L_0' L_0 \varphi_j = \varepsilon^{-1} \varphi_j, \quad j = 1, \dots r, \quad L_0' L_0 \varphi_j = \lambda_{j-r} \varphi_j, \quad j = r+1, \dots .$$

From Theorem 1 for $\varepsilon \le \lambda_1^{-1}$ we have

$$d^{N+r}(L(BH_{r,\varepsilon}), C(E)) \ge \sqrt{\sum_{j=N+1}^{\infty} \lambda_j}\,.$$

Since $T(BH) + X_r \supset L(BH_{r,\varepsilon})$

$$d^{N+r}(T(BH) + X_r, C(E)) \ge d^{N+r}(L(BH_{r,\varepsilon}), C(E)) \ge \sqrt{\sum_{j=N+1}^{\infty} \lambda_j}\,.$$

The equality

$$d^{N+r}(T(BH) + X_r, C(E)) = \lambda_{N+r}(T(BH) + X_r, C(E))$$

follows from the fact that H is a Hilbert space (compare with Proposition 8.8 [1, p. 33]). It is easy to show that

$$\lambda_{N+r}(T(BH) + X_r, C(E)) \le \lambda_N(T(BH), C(E)).$$

Now the upper bound follows directly from Theorem 1. The corollary is proved. ∎

Let H be a Hilbert space of functions defined on some set Ω. A function $K(z, w)$ defined on $\Omega \times \Omega$ is called a reproducing kernel of H if for each $w \in \Omega$, $K(z, w) \in H$ and for all $f \in H$

$$f(w) = (f(\cdot), K(\cdot, w))_H.$$

It is easily seen that

$$K(z, w) = \overline{K(w, z)}.$$

Let $E \subset \Omega$ be a compact with positive probability measure μ. Suppose that $Tf := f_{|E}$ is a bounded linear operator from H to $C(E)$.

THEOREM 2. *Let H and E be as above. Assume that $\varphi_1, \varphi_2, \ldots$ is a complete orthonormal basis for H and X_r is any r-dimensional subspace of $C(E)$ such that $X_r \perp H$ in $L_2(E, \mu)$. If $\varphi_1, \varphi_2, \ldots$ is an orthogonal system in $L_2(E, \mu)$ and $\lambda_j := \|\varphi_j\|^2_{L_2(E,\mu)}$ form a non-increasing sequence, then*

$$\sqrt{\sum_{j=N+1}^{\infty} \lambda_j} \le d^{N+r}(BH + X_r, C(E))$$

$$= \lambda_{N+r}(BH + X_r, C(E)) \le \sup_{z \in E} \sqrt{\sum_{j=N+1}^{\infty} |\varphi_j(z)|^2}.$$

Proof. Put $T_0 f := f_{|E}$. Let us consider T_0 as an operator from H into $L_2(E, \mu)$. For all $g \in L_2(E, \mu)$ we have

$$(T_0' g)(w) = ((T_0' g)(\cdot), K(\cdot, w))_H = (g(\cdot), T_0 K(\cdot, w))_{L_2(E,\mu)}$$

$$= \int_E g(z)\overline{K(z,w)}\, d\mu(z) = \int_E K(w,z) g(z)\, d\mu(z).$$

Thus the eigenvalue–eigenfunction problem

$$T_0' T_0 f = \lambda f$$

takes the form

$$\int_E K(w,z) f(z)\, d\mu(z) = \lambda f(w). \tag{2}$$

Since $\varphi_1, \varphi_2, \ldots$ is a complete orthonormal basis for H the representation

$$K(z,w) = \sum_{j=1}^{\infty} \varphi_j(z)\overline{\varphi_j(w)}$$

holds. In view of the orthogonality of the system $\varphi_1, \varphi_2, \ldots$ in $L_2(E, \mu)$ we have

$$\int_E K(w,z)\varphi_j(z)\, d\mu(z) = \lambda_j \varphi_j(w).$$

Thus λ_j is an eigenvalue and φ_j is an eigenfunction for Eq. (2). Now the theorem follows from Corollary 1. ∎

2. N-WIDTHS OF $H\mathscr{R}_2^r(B_n)$ AND $A\mathscr{R}_2^r(B_n)$

Set $N_m := \sum_{s=0}^{m-1}\binom{n+s-1}{n-1}$. Note that $N_m = \dim \mathscr{P}_{m-1}^n$, where $\mathscr{P}_m^n$ is the space of n-variable polynomials of degree m or less.

THEOREM 3.

(i) *For all* $0 < \rho < 1$ *and all* $m \geq r \geq 0$

$$d^{N_m}\big(H\mathscr{R}_2^r(B_n), C(S_\rho)\big) = \lambda_{N_m}\big(H\mathscr{R}_2^r(B_n), C(S_\rho)\big) = \rho^m\left(\frac{1}{(n-1)!}\sum_{s=0}^{\infty}\frac{((m-r+s)!)^2(n+m-1+s)!}{((m+s)!)^3}\rho^{2s}\right)^{1/2}.$$

(ii) *For all* $0 < \rho < 1$ *and all* $m \geq r \geq 1$

$$d^{N_m}\big(A\mathscr{R}_2^r(B_n), C(S_\rho)\big) = \lambda_{N_m}\big(A\mathscr{R}_2^r(B_n), C(S_\rho)\big) = \rho^m\left(\frac{1}{n!}\sum_{s=0}^{\infty}\frac{((m-r+s)!)^2(n+m+s)!}{((m+s)!)^3}\rho^{2s}\right)^{1/2}.$$

(iii) *For all*

$$0 < \rho \leq \left(\frac{n}{n+m}\right)^{1/(2m)}$$

$$d^{N_m}\big(BA_2(B_n), C(S_\rho)\big) = \lambda_{N_m}\big(BA_2(B_n), C(S_\rho)\big) = \rho^m\left(\frac{1}{n!}\sum_{s=0}^{\infty}\frac{(n+m+s)!}{(m+s)!}\rho^{2s}\right)^{1/2}$$
$$= \frac{\rho^m}{(1-\rho^2)^{(n+1)/2}}\left(\binom{n+m}{n}\sum_{s=0}^{n}\frac{(-1)^s}{1+s/m}\binom{n}{s}\rho^{2s}\right)^{1/2}. \tag{3}$$

Proof. For multiindex $\alpha := (\alpha_1, \ldots, \alpha_n)$ and $z \in \mathbb{C}^n$ set

$$z^\alpha := z_1^{\alpha_1}\cdots z_n^{\alpha_n}, \qquad |\alpha| := \alpha_1 + \cdots + \alpha_n, \qquad \alpha! := \alpha_1!\cdots\alpha_n!,$$
$$D_j := \partial/\partial z_j, \qquad D^\alpha := D_1^{\alpha_1}\cdots D_n^{\alpha_n}.$$

Denote by $\mathscr{H}_0$ the space of holomorphic functions in B_n for which $(D^\alpha f)(0) = 0$, $|\alpha| = 0, \dots, r-1$, and $\mathscr{R}^r f \in H_2(B_n)$. It is known (see [2]) that functions from $H_2(B_n)$ have finite boundary values almost everywhere. Moreover $H_2(B_n)$ can be considered as a Hilbert space with inner product

$$(f, g)_{H_2(B_n)} := \int_S f(z)\overline{g(z)}\, d\sigma(z).$$

Thus $\mathscr{H}_0$ is a Hilbert space with inner product

$$(f, g) := (\mathscr{R}^r f, \mathscr{R}^r g)_{H_2(B_n)}.$$

Let $f, g \in \mathscr{H}_0$ and

$$f(z) = \sum_{|\alpha|=r}^{\infty} c_\alpha z^\alpha, \qquad g(z) = \sum_{|\alpha|=r}^{\infty} d_\alpha z^\alpha.$$

Since monomials are orthogonal in $H_2(B_n)$ and

$$\|z^\alpha\|^2_{H_2(B_n)} = \frac{(n-1)!\alpha!}{(n-1+|\alpha|)!}$$

we have

$$(f, g) = \sum_{|\alpha|=r}^{\infty} \left(\frac{|\alpha|!}{(|\alpha| - r)!} \right)^2 \frac{(n-1)!\alpha!}{(n-1+|\alpha|)!} c_\alpha \bar{d}_\alpha .$$

It is easily verified that

$$K(z, w) = \sum_{|\alpha|=r}^{\infty} \left(\frac{(|\alpha| - r)!}{|\alpha|!} \right)^2 \frac{(n-1+|\alpha|)!}{(n-1)!\alpha!} \bar{w}^\alpha z^\alpha$$

is the reproducing kernel of $\mathscr{H}_0$.

Let us consider the space $L_2(S_\rho, \sigma_\rho)$, where σ_ρ is the probability measure on S_ρ which is invariant with respect to the orthogonal group $O(2n)$. Set for $|\alpha| \ge r$

$$\varphi_\alpha(z) := \frac{(|\alpha| - r)!}{|\alpha|!} \left(\frac{(n-1+|\alpha|)!}{(n-1)!\alpha!} \right)^{1/2} z^\alpha .$$

The functions $\varphi_\alpha(z)$ form a complete orthonormal basis for $\mathscr{H}_0$. Moreover these functions are orthogonal in $L_2(S_\rho, \sigma_\rho)$ and

$$\|\varphi_\alpha\|^2_{L_2(S_\rho,\sigma_\rho)} = \int_{S_\rho} |\varphi_\alpha(z)|^2 \, d\sigma_\rho(z)$$

$$= \int_S |\varphi_\alpha(\rho\xi)|^2 \, d\sigma(\xi) = \left(\frac{(|\alpha| - r)!}{|\alpha|!}\right)^2 \rho^{2|\alpha|}.$$

The number of different monomials z^α with $|\alpha| = s$ is equal to $\binom{n+s-1}{n-1}$. As $H\mathscr{R}_2^r(B_n) = B\mathscr{H}_0 + \mathscr{P}_r$, $\mathscr{H}_0 \perp \mathscr{P}_r$ in $L_2(S_\rho, \sigma_\rho)$, and $\dim \mathscr{P}_r = N_r$ we have by Theorem 2

$$\left(\sum_{s=m}^{\infty} \left(\frac{(s-r)!}{s!}\right)^2 \binom{n+s-1}{n-1} \rho^{2s}\right)^{1/2}$$

$$\le d^{N_m}\big(H\mathscr{R}_2^r(B_n), C(S_\rho)\big) = \lambda_{N_m}\big(H\mathscr{R}_2^r(B_n), C(S_\rho)\big)$$

$$\le \sup_{z \in S_\rho} \left(\sum_{|\alpha| \ge m} \left(\frac{(|\alpha| - r)!}{|\alpha|!}\right)^2 \frac{(n-1+|\alpha|)!}{(n-1)!\,\alpha!} |z^{2\alpha}|\right)^{1/2}.$$

Using the equation

$$\sum_{|\alpha|=s} \frac{s!}{\alpha!} |z^{2\alpha}| = |z|^{2s},$$

we obtain

$$d^{N_m}\big(H\mathscr{R}_2^r(B_n), C(S_\rho)\big)$$

$$= \lambda_{N_m}\big(H\mathscr{R}_2^r(B_n), C(S_\rho)\big) = \left(\sum_{s=m}^{\infty} \left(\frac{(s-r)!}{s!}\right)^2 \binom{n+s-1}{n-1} \rho^{2s}\right)^{1/2}$$

$$= \rho^m \left(\frac{1}{(n-1)!} \sum_{s=0}^{\infty} \frac{((m-r+s)!)^2 (n+m-1+s)!}{((m+s)!)^3} \rho^{2s}\right)^{1/2}.$$

To prove (ii) and (iii) we consider the space $\mathscr{A}_0$ of holomorphic functions in B_n for which $(D^\alpha f)(0) = 0$, $|\alpha| = 0, \dots, r-1$ and $\mathscr{R}^r f \in A_2(B_n)$. $\mathscr{A}_0$ is a Hilbert space with inner product

$$(f, g) := (\mathscr{R}^r f, \mathscr{R}^r g)_{A_2(B_n)} = \int_{B_n} \mathscr{R}^r f(z) \overline{\mathscr{R}^r g(z)} \, d\nu(z).$$

Analogous to the previous case, we can show that the functions

$$\psi_\alpha(z) := \sqrt{\frac{n + |\alpha|}{n}}\, \varphi_\alpha(z)$$

form a complete orthonormal basis for A_0 and orthogonal system in $L_2(S_\rho, \sigma_\rho)$. Furthermore

$$\|\psi_\alpha\|^2_{L_2(S_\rho, \sigma_\rho)} = \frac{n + |\alpha|}{n}\left(\frac{(|\alpha| - r)!}{|\alpha|!}\right)^2 \rho^{2|\alpha|} =: \lambda_{|\alpha|}.$$

Let $r \geq 1$ and $s \geq r$. Then

$$(n + s + 1)\left(\frac{s + 1 - r}{s + 1}\right)^2 \leq (n + s + 1)\left(\frac{s}{s + 1}\right)^2 \leq \frac{n + s + 1}{s + 1}s < n + s.$$

Thus

$$\lambda_{s+1} = \frac{n + s + 1}{n}\left(\frac{(s + 1 - r)!}{(s + 1)!}\right)^2 \rho^{2(s+1)} \leq \frac{n + s}{n}\left(\frac{(s - r)!}{s!}\right)^2 \rho^{2s} = \lambda_s.$$

If $r = 0$ (in this case $A\mathscr{R}_2^0(B_n) = BA_2(B_n)$), then $\{\lambda_j\}$ is not in general a non-increasing sequence. But if $((n + m)/n)\rho^{2m} \leq 1$ then for all $s \geq m$ and all $q < m$, $\lambda_q \geq \lambda_s$. Now (ii) and the first two equations of (3) follow from Theorem 2 in the same way as in the case of (i). Denote by

$$\Phi_n(m, \rho) := \sum_{s=0}^{\infty} \binom{n + m + s}{n} \rho^{2s}.$$

It easily verified that

$$\Phi_n(m, \rho) = \frac{1}{(1 - \rho^2)^{n+1}} \binom{n + m}{n} \sum_{s=0}^{n} \frac{(-1)^s}{1 + s/m} \binom{n}{s} \rho^{2s}.$$

So (iii) is proved. ∎

Remark. The referee informed me that in the case $n = 1$ the exact values of N-widths of the Bergman classes were obtained in [8].

For $n = 1$ the class $H\mathscr{R}_2^r(B_1)$ coincides with the class BH_2^r, defined as the set of all holomorphic functions in B_1 for which $f^{(r)}(z) \in BH_2(B_1)$. The set of all holomorphic functions in B_1 for which $f^{(r)}(z) \in BA_2(B_1)$ we denote by BA_2^r. If $r \geq 1$ the classes BA_2^r and $A\mathscr{R}_2^r(B_1)$ are different.

Nevertheless the method of Theorem 2 can be applied. Thus we obtain the following result.

THEOREM 4. *Let* $0 < \rho < 1$. *Then*:

(i) *for all* $N \geq r \geq 0$

$$d^N\big(BH_2^r, C(S_\rho)\big) = \lambda_N\big(BH_2^r, C(S_\rho)\big) = \rho^N\left(\sum_{s=0}^{\infty}\left(\frac{(N-r+s)!}{(N+s)!}\right)^2 \rho^{2s}\right)^{1/2};$$

(ii) *for all* $N \geq r \geq 1$

$$d^N\big(BA_2^r, C(S_\rho)\big) = \lambda_N\big(BA_2^r, C(S_\rho)\big) = \rho^N\left(\sum_{s=0}^{\infty}\left(\frac{(N-r+s)!}{(N+s)!}\right)^2 (N+s+1)\rho^{2s}\right)^{1/2}.$$

3. THE N-WIDTHS FOR HARDY AND BERGMAN CLASSES IN POLYDISCS

Set

$$U^n := \{z \in \mathbb{C}^n : |z_1| < 1, \dots, |z_n| < 1\},$$
$$T^n := \{z \in \mathbb{C}^n : |z_1| = 1, \dots, |z_n| = 1\},$$
$$T_\rho^n := \{z \in \mathbb{C}^n : |z_1| = \rho_1, \dots, |z_n| = \rho_n\},$$

where $\rho = (\rho_1, \dots, \rho_n)$ and $0 \leq \rho_j < 1$, $j = 1, \dots, n$. Denote by $H_2(U^n)$ the set of all holomorphic functions in U^n for which

$$\|f\|_{H_2(U^n)} := \sup_{0<r<1}\left(\int_{T^n} |f(rz)|^2\, d\mu(z)\right)^{1/2} < \infty,$$

where $\mu(z)$ is the normalized Lebesgue measure in T^n. We shall denote by $A_2(U^n)$ the set of all holomorphic functions in U^n for which

$$\|f\|_{A_2(U^n)} := \left(\int_{U^n} |f(z)|^2\, d\omega(z)\right)^{1/2} < \infty,$$

where $\omega(z)$ is the normalized Lebesgue measure in U^n. The spaces

$H_2(U^n)$ and $A_2(U^n)$ are Hilbert spaces with the reproducing kernels

$$K_H(z,w) := \begin{cases} (1 - z_1\bar{w}_1)^{-1} \cdots (1 - z_n\bar{w}_n)^{-1}, & H = H_2(U^n), \\ (1 - z_1\bar{w}_1)^{-2} \cdots (1 - z_n\bar{w}_n)^{-2}, & H = A_2(U^n) \end{cases}$$

(the details can be found in [9]).

THEOREM 5. *Let* $\rho = (\rho_1, \ldots, \rho_n)$, $0 \le \rho_j < 1$.

(i) *Assume that* $\alpha^{(1)}, \ldots, \alpha^{(N)}$ *are the N largest terms of the sequence* $\{\rho^{2\alpha}\}$. *Then*

$$d^N\big(BH_2(U^n), C(T_\rho^n)\big) = \lambda_N\big(BH_2(U^n), C(T_\rho^n)\big)$$
$$= \left((1 - \rho_1^2)^{-1} \cdots (1 - \rho_n^2)^{-1} - \sum_{s=1}^{N} \rho^{2\alpha^{(s)}}\right)^{1/2}. \tag{4}$$

(ii) *Assume that* $\alpha^{(1)}, \ldots, \alpha^{(N)}$ *are the N largest terms of the sequence* $\{k_\alpha \rho^{2\alpha}\}$, *where* $k_\alpha := (\alpha_1 + 1) \cdots (\alpha_n + 1)$. *Then*

$$d^N\big(BA_2(U^n), C(T_\rho^n)\big) = \lambda_N\big(BA_2(U^n), C(T_\rho^n)\big)$$
$$= \left((1 - \rho_1^2)^{-2} \cdots (1 - \rho_n^2)^{-2} - \sum_{s=1}^{N} k_{\alpha^{(s)}} \rho^{2\alpha^{(s)}}\right)^{1/2}.$$

Proof. Let us prove (i). The monomials z^α form a complete orthonormal basis in $H_2(U^n)$. They are also an orthogonal system in $L_2(T_\rho^n, \mu_\rho)$, where μ_ρ is the normalized Lebesgue measure in T_ρ^n. Moreover

$$\|z^\alpha\|^2_{L_2(T_\rho^n, \mu_\rho)} = \rho^{2\alpha}$$

and for $z \in T_\rho^n$, $|z^\alpha|^2 = \rho^{2\alpha}$. From Theorem 2 we have

$$d^N\big(BH_2(U^n), C(T_\rho^n)\big) = \lambda_N\big(BH_2(U^n), C(T_\rho^n)\big) = \left(\sum_{\alpha \notin \tau} \rho^{2\alpha}\right)^{1/2},$$

where $\tau := \{\alpha^{(1)}, \ldots, \alpha^{(N)}\}$. Now (i) follows from the representation

$$(1 - \rho_1^2)^{-1} \cdots (1 - \rho_n^2)^{-1} = \sum_{|\alpha| \ge 0} \rho^{2\alpha}.$$

Using the representation

$$\left(1-\rho_1^2\right)^{-2}\cdots\left(1-\rho_n^2\right)^{-2}=\sum_{|\alpha|\geq 0}k_\alpha\rho^{2\alpha}, \tag{5}$$

a similar argument proves (ii). ∎

We can obtain a more precise result in the case $\rho_1=\cdots=\rho_n$.

THEOREM 6. *Let* $\rho_1=\cdots=\rho_n=\rho$ *and* $0<\rho<1$. *Then*:

(i) *for* $N_{m-1}<N\leq N_m$

$$\begin{aligned} d^N\left(BH_2(U^n),C(T_\rho^n)\right) &= \lambda_N\left(BH_2(U^n),C(T_\rho^n)\right)\\ &= \rho^{m-1}\left(N_m-N+\binom{n+m-1}{n-1}\left(1-\rho^2\right)^{-n}\right.\\ &\qquad\left.\times\sum_{s=0}^{n-1}\frac{(-1)^s}{1+s/m}\binom{n-1}{s}\rho^{2(s+1)}\right)^{1/2}; \end{aligned}$$

(ii) *for* $n\geq 2$ *and*

$$0<\rho\leq m^{1/2}(m/n+1)^{-n/2} \tag{6}$$

$$\begin{aligned} &d^{N_m}\left(BA_2(U^n),C(T_\rho^n)\right)\\ &\quad=\lambda_{N_m}\left(BA_2(U^n),C(T_\rho^n)\right)\\ &\quad=\frac{\rho^m}{(1-\rho^2)^n}\left(\binom{2n+m-1}{2n-1}\sum_{s=0}^{2n-1}\frac{(-1)^s}{1+s/m}\binom{2n-1}{s}\rho^{2s}\right)^{1/2}. \end{aligned}$$

Proof. The sequence $\rho^{2|\alpha|}$ is a non-increasing sequence for $|\alpha|\to\infty$. The number of different multiindexes α with $|\alpha|=s$ is equal to $\binom{n+s-1}{n-1}$. By (4) we have for $N_{m-1}<N\leq N_m$

$$\begin{aligned} &d^N\left(BH_2(U^n),C(T_\rho^n)\right)\\ &\quad=\lambda_N\left(BH_2(U^n),C(T_\rho^n)\right)\\ &\quad=\left(\left(1-\rho^2\right)^{-n}-\sum_{s=0}^{m-2}\binom{n+s-1}{n-1}\rho^{2s}-(N-N_{m-1})\rho^{2(m-1)}\right)^{1/2}\\ &\quad=\left((N_m-N)\rho^{2(m-1)}+\sum_{s=m}^{\infty}\binom{n+s-1}{n-1}\rho^{2s}\right)^{1/2}. \end{aligned}$$

Now (i) follows from equations

$$\sum_{s=m}^{\infty} \binom{n+s-1}{n-1} \rho^{2s}$$

$$= \rho^{2m} \sum_{s=0}^{\infty} \binom{n+m+s-1}{n-1} \rho^{2s} = \rho^{2m} \Phi_{n-1}(m, \rho)$$

$$= \rho^{2m}(1-\rho^2)^{-n} \binom{n+m-1}{n-1} \sum_{s=0}^{n-1} \frac{(-1)^s}{1+s/m} \binom{n-1}{s} \rho^{2s}.$$

To prove (ii) we will first prove that if the condition (6) holds, then for all $|\beta| \geq m$ and all $|\alpha| < m$

$$k_\beta \rho^{2|\beta|} \leq k_\alpha \rho^{2|\alpha|}. \tag{7}$$

In view of the monotone decreasing property of $y(x) := x(x/n+1)^{-n}$ for $x \geq 2$ and $n \geq 2$ we have

$$\rho^2 \leq \max\{y(1), y(2)\} \leq 1/2.$$

Consequently for all $s \geq 1$

$$(s+1)\rho^{2s} \leq s\rho^{2s-2}.$$

Thus for each $|\beta| \geq m$ choosing any $\beta_j \geq 1$ we will have

$$k_\beta \rho^{2|\beta|} \leq k_{\beta'} \rho^{2|\beta'|},$$

where $\beta' = (\beta_1, \ldots, \beta_j - 1, \ldots, \beta_n)$. Continuing this process we will find β^* with $|\beta^*| = m$ for which

$$k_\beta \rho^{2|\beta|} \leq k_{\beta^*} \rho^{2|\beta^*|} \leq (m/n+1)^n \rho^{2m}.$$

On the other hand, if $|\alpha| < m$ then in view of the monotone decreasing of the sequence $\{s\rho^{2s-2}\}_1^\infty$ and by (6) we obtain

$$k_\alpha \rho^{2|\alpha|} \geq (|\alpha|+1)\rho^{2|\alpha|} \geq m\rho^{2m-2} \geq (m/n+1)^n \rho^{2m}.$$

So (7) is proved.

From Theorem 5 it follows that

$$d^{N_m}(BA_2(U^n), C(T_\rho^n)) = \lambda_{N_m}(BA_2(U^n), C(T_\rho^n))$$

$$= \left((1-\rho^2)^{-2n} - \sum_{|\alpha|=0}^{m-1} k_\alpha \rho^{2|\alpha|}\right)^{1/2} =: d.$$

By (5)

$$\sum_{|\alpha|=s} k_\alpha = \binom{2n+s-1}{2n-1}.$$

Therefore

$$\begin{aligned} d^2 &= \sum_{s=m}^{\infty} \binom{2n+s-1}{2n-1} \rho^{2s} = \rho^{2m} \sum_{s=0}^{\infty} \binom{2n+s+m-1}{2n-1} \rho^{2s} \\ &= \rho^{2m} \Phi_{2n-1}(m, \rho) \\ &= \rho^{2m}(1-\rho^2)^{-2n} \binom{2n+m-1}{2n-1} \sum_{s=0}^{2n-1} \frac{(-1)^s}{1+s/m} \binom{2n-1}{s} \rho^{2s}. \end{aligned}$$

4. N-Widths of Holomorphic Functions of One Variable

Denote by H_γ the space of holomorphic functions in Δ_R

$$f(z) = \sum_{s=-\infty}^{+\infty} a_s z^s$$

which satisfy the condition

$$\sum_{s=-\infty}^{+\infty} \gamma_s |a_s|^2 < \infty,$$

where $\{\gamma_s\}$ is a sequence of non-negative numbers such that $\liminf_{s \to \mp\infty} \gamma_s^{1/|s|} \geq R^2$. Set $\Gamma := \{s \colon \gamma_s = 0\}$ and $r :=$ card Γ.

The space

$$H_\gamma^0 := \left\{ f(z) = \sum_{s=-\infty}^{+\infty} a_s z^s \in H_\gamma \colon a_j = 0, j \in \Gamma \right\}$$

is a Hilbert space with inner product

$$(f, g) = \sum_{s=-\infty}^{+\infty} \gamma_s a_s \bar{b}_s,$$

where

$$f(z) = \sum_{s=-\infty}^{+\infty} a_s z^s, \qquad g(z) = \sum_{s=-\infty}^{+\infty} b_s z^s.$$

Moreover the space H_γ^0 has the reproducing kernel

$$K(z,w) := \sum_{s \notin \Gamma} \gamma_s^{-1} z^s \overline{w}^s.$$

Set $BH_\gamma := BH_\gamma^0 + \mathscr{P}_r$, where $\mathscr{P}_r := \{\Sigma_{s \in \Gamma} a_s z^s\}$. This convenient form for generalization of certain classes in the case of the unit disk was proposed by Fisher and Micchelli [10].

For $1 \le \rho < R$ and $k \ge r$ set $\sigma_k(\rho) := \{s_1, \dots, s_{k-r}\} \cup \Gamma$, where $\{s_1, \dots, s_{k-r}\}$ are the $k - r$ largest terms of the sequence

$$\left\{ \gamma_s^{-1} \frac{\rho^{2s} + \rho^{-2s}}{2} \right\}_{s \notin \Gamma}.$$

THEOREM 7. *Assume that for all* $s \in \mathbb{N}$ $\gamma_s = \gamma_{-s}$.

(i) *If* $N \ge (r+1)/2$ *and* $0 \in \sigma_{2N-1}(\rho)$, *then*

$$d^{2N-1}(BH_\gamma, C(\Delta_\rho)) = \lambda_{2N-1}(BH_\gamma, C(\Delta_\rho)) = \left(\sum_{s \notin \sigma_{2N-1}(\rho)} \gamma_s^{-1} \frac{\rho^{2s} + \rho^{-2s}}{2} \right)^{1/2}.$$

(ii) *If* $N \ge r/2$ *and* $0 \notin \sigma_{2N}(\rho)$, *then*

$$d^{2N}(BH_\gamma, C(\Delta_\rho)) = \lambda_{2N}(BH_\gamma, C(\Delta_\rho)) = \left(\gamma_0 + \sum_{s \notin \sigma_{2N}(\rho)} \gamma_s^{-1} \frac{\rho^{2s} + \rho^{-2s}}{2} \right)^{1/2}.$$

Proof. Let us prove (i). The functions

$$\varphi_s(z) := \gamma_s^{-1/2} z^s, \qquad s \notin \Gamma$$

form a complete orthonormal basis for H_γ^0. Denote by $L_2(\partial\Delta_\rho)$ a Hilbert space of functions defined on the boundary of Δ_ρ with inner product

$$(f,g) := \frac{1}{4\pi} \int_0^{2\pi} \left[f(\rho e^{i\theta}) \overline{g(\rho e^{i\theta})} + f(\rho^{-1} e^{i\alpha}) \overline{g(\rho^{-1} e^{i\theta})} \right] d\theta.$$

It is easily seen that φ_s form an orthogonal system in $L_2(\partial\Delta_\rho)$ and

$$\|\varphi_s\|^2_{L_2(\partial\Delta_\rho)} = \gamma_s^{-1}\frac{\rho^{2s}+\rho^{-2s}}{2}.$$

From Theorem 2 follows

$$\begin{aligned}\left(\sum_{s\notin\sigma_{2N-1}(\rho)}\gamma_s^{-1}\frac{\rho^{2s}+\rho^{-2s}}{2}\right)^{1/2}\\ &\le d^{2N-1}\big(BH_\gamma, C(\Delta_\rho)\big)\\ &= \lambda_{2N-1}\big(BH_\gamma, C(\Delta_\rho)\big) \le \sup_{z\in\partial\Delta_\rho}\left(\frac{1}{2}\sum_{s\notin\sigma_{2N-1}(\rho)}\gamma_s^{-1}(|z|^s+|z|^{-s})\right)^{1/2}\\ &= \left(\sum_{s\notin\sigma_{2N-1}(\rho)}\gamma_s^{-1}\frac{\rho^{2s}+\rho^{-2s}}{2}\right)^{1/2}.\end{aligned}$$

Part (ii) is proved in a similar way. ∎

For $\rho = 1$ the analogous application of Theorem 2 gives

THEOREM 8. *For all* $N \ge r$

$$d^N\big(BH_\gamma, C(\Delta_1)\big) = \lambda_N\big(BH_\gamma, C(\Delta_1)\big) = \left(\sum_{s\notin\sigma_N(1)}\gamma_s^{-1}\right)^{1/2}.$$

Now we consider some examples of the spaces H_γ. Denote by $H_2(\Delta_R)$ the class of holomorphic functions in Δ_R for which

$$\|f\|_{H_2(\Delta_R)} := \sup_{1<\rho<R}\left(\frac{1}{4\pi}\int_0^{2\pi}\Big[|f(\rho e^{i\theta})|^2 + |f(\rho^{-1}e^{i\theta})|^2\Big]\,d\theta\right)^{1/2} < \infty.$$

Let $A_2(\Delta_R)$ be the class of holomorphic functions in Δ_R for which

$$\|f\|_{A_2(\Delta_R)} := \left(\int_{\Delta_R}|f(z)|^2\,d\eta(z)\right)^{1/2} < \infty,$$

where $\eta(z)$ is normalized Lebesgue measure in Δ_R. Let us consider the classes $BH_2^r(\Delta_R)$ and $BA_2^r(\Delta_R)$, which are the sets of holomorphic functions in Δ_R such that $f^{(r)}(z)$ lies in $BH_2(\Delta_R)$ and $BA_2(\Delta_R)$, respectively.

It can be easily shown that the class $BH_2^r(\Delta_R)$ coincides with BH_γ for

$$\gamma_s = (s(s-1)\cdots(s-r+1))^2 \frac{R^{2(s-r)} + R^{-2(s-r)}}{2},$$

and $BA_2^r(\Delta_R)$ coincides with BH_γ, where for $r \geq 1$

$$\gamma_s = (s(s-1)\cdots(s-r+2))^2(s-r+1)\frac{R^{2(s-r+1)} + R^{-2(s-r+1)}}{R^2 - R^{-2}}$$

and for $r = 0$ (that is for $BA_2(\Delta_R)$)

$$\gamma_s = (s+1)^{-1}\frac{R^{2(s+1)} + R^{-2(s+1)}}{R^2 - R^{-2}}, \qquad s \neq -1, \qquad \gamma_{-1} = \frac{4\log R}{R^2 - R^{-2}}.$$

We give some more examples of the classes BH_γ. Let $H_2(D_H)$ and $A_2(D_H)$ be the sets of all 2π-periodic holomorphic functions in D_H which satisfy the conditions

$$\|f\|_{H_2(D_H)} := \sup_{0<h<H}\left(\frac{1}{4\pi}\int_0^{2\pi}\left[|f(x+ih)|^2 + |f(x-ih)|^2\right]dx\right)^{1/2} < \infty$$

and

$$\|f\|_{A_2(D_H)} := \left(\frac{1}{4\pi H}\int_0^{2\pi}\int_{-H}^{H}|f(x+iy)|^2\,dx\,dy\right)^{1/2} < \infty,$$

respectively. Denote by $BH_2^r(D_H)$ and $BA_2^r(D_H)$ the sets of all 2π-periodic holomorphic functions in D_H for which $f^{(r)}(z)$ lie in $BH_2(D_H)$ and $BA_2(D_H)$, respectively.

To find the linear and Gel'fand N-widths of $BH_2^r(D_H)$ and $BA_2^r(D_H)$ in the space $C(D_h)$, $0 \leq h < H$, we use the map $z = (1/i)\log w$. Then the original problem reduces to the one for BH_γ with $R = e^H$ and the space $C(\Delta_\rho)$ with $\rho = e^h$, where

$$\gamma_s = s^{2r}\cosh(2sH)$$

in the case of $BH_2^r(D_H)$ and

$$\gamma_s = \frac{1}{2H}s^{2r-1}\sinh(2sH)$$

in the case of $BA_2^r(D_H)$.

By Theorems 7 and 8 we obtain the following result.

THEOREM 9. *Let* $r \geq 0$.

(i) *For all* $0 \leq h < H$

$$d^{2N-1}(BH_2^r(D_H), C(D_h)) = \lambda_{2N-1}(BH_2^r(D_H), C(D_h)) = \left(2 \sum_{s=N}^{\infty} \frac{\cosh(2sh)}{s^{2r}\cosh(2sH)}\right)^{1/2},$$

$$d^{2N-1}(BA_2^r(D_H), C(D_h)) = \lambda_{2N-1}(BA_2^r(D_H), C(D_h)) = 2H^{1/2}\left(\sum_{s=N}^{\infty} \frac{\cosh(2sh)}{s^{2r-1}\sinh(2sH)}\right)^{1/2}.$$

(ii) *For all* $H > 0$

$$d^{2N}(BH_2^r(D_H), C[0,2\pi]) = \lambda_{2N}(BH_2^r(D_H), C[0,2\pi]) = \left(\frac{1}{N^{2r}\cosh(2NH)} + 2\sum_{s=N+1}^{\infty} \frac{1}{s^{2r}\cosh(2sH)}\right)^{1/2},$$

$$d^{2N}(BA_2^r(D_H), C[0,2\pi]) = \lambda_{2N}(BA_2^r(D_H), C[0,2\pi]) = H^{1/2}\left(\frac{2}{N^{2r-1}\sinh(2NH)} + 4\sum_{s=N+1}^{\infty} \frac{1}{s^{2r-1}\sinh(2sH)}\right)^{1/2}.$$

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